Tutorial 4.

Quadratic Variation

Define: The quadratic variation of function f up to time T is

$$[f,f](T) = \lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} [f(t_{j+1}) - f(t_j)]^2, \quad \text{where } \Pi = \{t_0, t_1, \dots, t_n\}, \ 0 \le t_0 < t_1 < \cdots < t_n =$$

Theorem: Let W be a Brownian motion. Then $[W,W](T) = T$ for all $T \geq 0$ almost surely.

Proof: By the definition, $[W,W](T) = \lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} [W(t_{j+1}) - W(t_j)]^2$, $\|\Pi\| = \max\{|t_1 - t_0|, |t_2 - t_1|, \dots\}$.

Set $Q_\Pi = \sum_{j=0}^{n-1} (W(t_{j+1}) - W(t_j))^2$, and then

$$E Q_\Pi = \sum_{j=0}^{n-1} E(W(t_{j+1}) - W(t_j))^2,$$

Since $W(t_{j+1}) - W(t_j) \sim N(0, t_{j+1} - t_j)$, then

$$E(W(t_{j+1}) - W(t_j))^2 = Var(W(t_{j+1}) - W(t_j)) = t_{j+1} - t_j.$$

The sum $\sum_{j=0}^{n-1} E(W(t_{j+1}) - W(t_j))^2 = \sum_{j=0}^{n-1} t_{j+1} - t_j = T.$

Hence, $\lim_{\|\Pi\|\to 0} E Q_\Pi = T.$

On the other hand,

$$Var[W(t_{j+1}) - W(t_j)]^2 = E[(W(t_{j+1}) - W(t_j))^4] - 2(t_{j+1} - t_j)\, E[(W(t_{j+1}) - W(t_j))^2] + (t_{j+1} - t_j)^2$$

where $E[(W(t_{j+1}) - W(t_j))]^4 = 3(t_{j+1} - t_j)^2$. as $MGF = e^{\theta^2 (t_{j+1} - t_j)/2}$

then $\mathrm{Var}[(W(t_{j+1})-W(t_j))^2] = 3(t_{j+1}-t_j)^2 - 2(t_{j+1}-t_j)^2 + (t_{j+1}-t_j)^2$

$$= 2(t_{j+1}-t_j)^2$$

since $W_{t_{j+1}} - W_{t_j} \perp W_{t_j} - W_{t_{j-1}}$.

and $\mathrm{Var}(Q_\Pi) = \sum_{j=0}^{n-1} \mathrm{Var}[(W_{t_{j+1}} - W(t_j))^2] = \sum_{j=0}^{n-1} 2(t_{j+1}-t_j)^2 \le \sum_{j=0}^{n-1} 2\|\Pi\|(t_{j+1}-t_j)$

$$= 2\|\Pi\| \cdot T.$$

hence, $\lim_{\|\Pi\|\to 0} \mathrm{Var}(Q_\Pi) = 0$. Then we can conclude

$\lim_{\|\Pi\|\to 0} Q_\Pi = EQ_\Pi = T$. #

Remark: In addition to quadratic variation, the cross variation of $W(t)$ and t, and quadratic variation of t are

$\lim_{\|\Pi\|\to 0} \sum_{j=1}^{n-1} (W(t_{j+1}) - W(t_j))(t_{j+1}-t_j) = 0$. and $\lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} (t_{j+1} - t_j)^2 = 0$

Proof: $|W(t_{j+1}) - W(t_j)(t_{j+1}-t_j)| \le \max_{0\le k\le n-1} |W(t_{k+1}) - W(t_k)| (t_{j+1}-t_j)$

$$\le \max_{0\le k\le n-1} |W(t_{k+1}) - W(t_k)| \cdot T$$

Since W is continous, $\lim_{\|\Pi\|\to 0} \max |W(t_{k+1}) - W(t_k)| = 0$.

then $\lim_{\|\Pi\|\to 0} |W(t_{j+1}) - W(t_j)(t_{j+1}-t_j)| = 0$.

$\sum_{j=1}^{n-1} (t_{j+1} - t_j)^2 \le \max_{0\le k\le n-1} (t_{k+1}-t_k) \cdot \sum_{j=0}^{n-1} (t_{j+1}-t_j) = \|\Pi\| \cdot T$

then $\cancel{\lim_{\|\Pi\|\to 0} |W(t_{j+1}) - W(t_j)(t_{j+1}-t_j)|}$ $\lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} (t_{j+1}-t_j)^2 = 0$. #

Write $dW(t)dW(t) = dt$, $dW(t)dt = 0$, $dtdt = 0$.